

ON LOEWNER'S CHARACTERIZATION OF POLYNOMIALS

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ABSTRACT. We give a new demonstration of Loewner's characterization of polynomials, solving in the positive a conjecture proposed by Laird and McCann in 1984.

1. INTRODUCTION

We give a new proof of the following result:

Theorem 1.1 (Loewner, 1959). *Assume that $d > 1$ is a natural number. Let $f \in C(\mathbb{R}^d)$ and let $R_f = \text{span}\{f(Lx) : L \text{ is an isometry of } \mathbb{R}^d\}$. Then f is an ordinary polynomial if and only if $\dim R_f < \infty$.*

This theorem was demonstrated by Loewner using Anselone-Koreevar's theorem [5], which claims that exponential polynomials can be characterized as elements of finite dimensional translation invariant subspaces of $C(\mathbb{R}^d)$. Later on, in 1984, Laird and McCann [8] gave a characterization of polynomials as the elements of finite dimensional subspaces of $C(\mathbb{R}^d)$ which are simultaneously translation and dilation invariant (their result allows $d = 1$). Their proof used Fréchet's characterization of polynomials, instead of Anselone-Koreevar's theorem. Then they conjectured that a similar proof for Loewner's theorem should exist. The main goal of this short note is to solve their conjecture in the positive.

2. MAIN RESULT

Note that, if $\dim R_f < \infty$ then for all $y \in \mathbb{R}^d$ and all $P \in \mathbf{O}(d) := \{A \in \mathbf{GL}_d(\mathbb{R}) : A^t = A^{-1}\}$ the maps $\tau_y(g)(x) = g(x + y)$ and $O_P(g)(x) = g(Px)$ are well defined as operators from R_f into R_f . In fact, they are automorphisms of this space, since they are injective and $\dim R_f < \infty$. Indeed, $(\tau_y)^{-1} = \tau_{-y}$ and $(O_P)^{-1} = O_{P^{-1}} = O_{Pt}$.

Let X_d denote indistinctly either the space $\mathcal{D}(\mathbb{R}^d)'$ of Schwartz complex valued distributions defined on \mathbb{R}^d or the space $C(\mathbb{R}^d)$ of continuous complex valued functions defined on \mathbb{R}^d . For $f \in \mathcal{D}(\mathbb{R}^d)'$ we can introduce the translation operator

$$\tau_h(f)\{\phi\} = f\{\tau_{-h}(\phi)\}, \text{ where } h \in \mathbb{R}^d \text{ and } \phi \in \mathcal{D}(\mathbb{R}^d) \text{ is any test function,}$$

and the operator

$$O_P(f)\{\phi\} = \frac{1}{|\det(P)|} f\{O_{P^{-1}}(\phi)\},$$

where $P \in \mathbf{GL}_d(\mathbb{R})$ is any invertible matrix, $\phi \in \mathcal{D}(\mathbb{R}^d)$ is any test function, and $O_{P^{-1}}(\phi)(x) = \phi(P^{-1}x)$ for all $x \in \mathbb{R}^d$.

We demonstrate the following result, which is stronger than Loewner's theorem, since it works for distributions.

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Theorem 2.1. *Let $d \geq 2$ be a natural number, let $f \in X_d$ and assume that, for a certain finite dimensional space $V \subseteq X_d$ we have that:*

- $f \in V$
- V is translation invariant (i.e., $\tau_h(V) \subseteq V$ for all $h \in \mathbb{R}^d$).
- V is invariant by orthogonal transformations of \mathbb{R}^d (i.e., $O_P(V) \subseteq V$, for all $P \in \mathbf{O}(d)$).

Then f is, in distributional sense, an ordinary polynomial on \mathbb{R}^d . In particular, f is equal almost everywhere to an ordinary polynomial and, if f is a continuous ordinary function, then it is an ordinary polynomial.

Our proof is based on the following theorem

Theorem 2.2. *Let f be a complex valued distribution defined on \mathbb{R}^d . Assume that $q(\tau_y)f = 0$ for all y with $\|y\| \leq \delta$, for a certain polynomial $q(z) = a_0 + a_1z + \cdots + a_nz^n$ such that $a_0 \neq 0$. Then f is, in distributional sense, an ordinary polynomial.*

Proof. By assumption, $q(\tau_y)f = 0$ for all y with $\|y\| \leq \delta$, which means that

$$(1) \quad 0 = \sum_{k=0}^n a_k \tau_{ky} f(x) \text{ for all } y \in B_d(\delta) := \{h \in \mathbb{R}^d : \|h\| < \delta\}.$$

Assume that $y, h_1 \in B_d(\delta/2)$ (so that $y^* = y - h_1 \in B_d(\delta)$) and use (1) with y^* to conclude that:

$$(2) \quad 0 = \sum_{k=0}^n a_k \tau_{ky^*} f(x) = \sum_{k=0}^n a_k \tau_{ky-kh_1} f(x) \text{ for all } y, h_1 \in B_d(\delta/2).$$

Apply τ_{nh_1} to both sides of the equation. Then

$$(3) \quad 0 = \sum_{k=0}^n a_k \tau_{nh_1} \tau_{ky-kh_1} f(x) = \sum_{k=0}^n a_k \tau_{(n-k)h_1} (\tau_{ky} f)(x) \text{ for all } y, h_1 \in B_d(\delta/2).$$

Taking differences between (3) and (1), we conclude that

$$(4) \quad 0 = \sum_{k=0}^{n-1} a_k \Delta_{(n-k)h_1} (\tau_{ky} f)(x) \text{ for all } y, h_1 \in B_d(\delta/2).$$

We can repeat the argument, reducing the norm of y, h_1 to $\delta/4, \delta/8$, etc., which leads to the equation

$$a_0 \Delta_{h_n} \Delta_{2h_{n-1}} \cdots \Delta_{(n-1)h_2} \Delta_{nh_1} (f)(x) = 0 \text{ for all } h_1, \dots, h_n \in B_d(\delta/2^n).$$

The result follows from Montel's type version of Fréchet's theorem for distributions, since $a_0 \neq 0$ by hypothesis (see, e.g., [1, 2, 3, 4]). \square

We also use the following technical result, which is well known (see, for example, [7, 11, 12] and [6], for the original exposition of this result):

Lemma 2.3 (Frobenius, 1896). *Let V be a finite dimensional complex vector space. Assume that $T, S : V \rightarrow V$ are commuting linear operators (i.e., $TS = ST$). Then they are simultaneously triangularizable. In particular, the eigenvalues $\lambda_i(T), \lambda_i(S)$ and $\lambda_i(TS)$ of T, S and TS , respectively, can be arranged, counting multiplicities, in such a way that $\lambda_i(TS) = \lambda_i(T)\lambda_i(S)$ for $i = 1, \dots, \dim V$.*

Proof of Theorem 2.1. Obviously

$$\tau_{y+z} = \tau_y \tau_z = \tau_z \tau_y \text{ for all } y, z \in \mathbb{R}^d,$$

and

$$(5) \quad \tau_{Py} = (O_P)^{-1} \tau_y O_P = O_{Pt} \tau_y O_P \text{ for all } y \in \mathbb{R}^d.$$

To demonstrate (5) we consider first the case of ordinary functions. In that setting it is clear that $(O_P)^{-1} = O_{(P)^{-1}} = O_{P^t}$ and

$$\begin{aligned} (O_{P^{-1}} \tau_y O_P g)(x) &= (O_{P^{-1}} \tau_y (g(Px))) = O_{P^{-1}} (g(P(x+y))) \\ &= O_{P^{-1}} (g(Px + Py)) = g(P^{-1}Px + Py) \\ &= g(x + Py) = \tau_{Py}(g)(x), \end{aligned}$$

for every function g . Assume now that V is a space of distributions and $g \in V$. Then, for every test function ϕ ,

$$\begin{aligned} (O_{P^{-1}} \tau_y O_P)(g)\{\phi\} &= O_{P^{-1}}(\tau_y O_P(g))\{\phi\} = \frac{1}{|\det(P^{-1})|} (\tau_y O_P(g))\{O_P(\phi)\} \\ &= \frac{1}{|\det(P^{-1})|} \tau_y(O_P(g))\{O_P(\phi)\} = \frac{1}{|\det(P^{-1})|} (O_P(g))\{\tau_{-y} O_P(\phi)\} \\ &= \frac{1}{|\det(P^{-1})|} \frac{1}{|\det(P)|} g\{O_{P^{-1}} \tau_{-y} O_P(\phi)\} = g\{O_{P^{-1}} \tau_{-y} O_P(\phi)\} \\ &= g\{\tau_{-Py}(\phi)\} = \tau_{Py}(g)\{\phi\}, \end{aligned}$$

which is what we wanted to prove. It follows that

- (a) For any $y \in \mathbb{R}^d$, the operators $\tau_y : V \rightarrow V$ and $\tau_{Py} : V \rightarrow V$ have the very same eigenvalues and characteristic polynomial.
- (b) For any $y, z \in \mathbb{R}^d$, the operators $\tau_y : V \rightarrow V$ and $\tau_z : V \rightarrow V$ are simultaneously triangularizable, since they are commuting operators (i.e., $\tau_y \tau_z = \tau_z \tau_y$). In particular, it is possible to arrange the eigenvalues $\lambda_i(\tau_y)$, $\lambda_i(\tau_z)$ and $\lambda_i(\tau_{y+z})$ of τ_y , τ_z and τ_{y+z} , respectively, in such a way that

$$\lambda_i(\tau_{y+z}) = \lambda_i(\tau_y) \lambda_i(\tau_z), \quad i = 1, \dots, N := \dim V.$$

Let $y \in \mathbb{R}^d$. From (a) we have that, for a certain permutation σ of $\{1, \dots, N\}$ (which depends on P),

$$\lambda_i(\tau_{Py}) = \lambda_{\sigma(i)}(\tau_y), \quad i = 1, \dots, N.$$

Moreover, if we set $z = (P - I)y$, then $Py = y + (P - I)y$ and (b) implies that:

$$\lambda_{\sigma(i)}(\tau_y) = \lambda_i(\tau_{Py}) = \lambda_i(\tau_y) \lambda_i(\tau_{(P-I)y}), \quad i = 1, \dots, N.$$

Hence

$$\lambda_i(\tau_{(P-I)y}) = \frac{\lambda_{\sigma(i)}(\tau_y)}{\lambda_i(\tau_y)}, \quad i = 1, \dots, N,$$

since τ_y is an automorphism, which implies that $\lambda_i(\tau_y) \neq 0$ for all i .

Take $y = e_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$. Let us consider the polynomial

$$q(z) = \prod_{\sigma \in S_N} \prod_{i=1}^N \left(z - \frac{\lambda_{\sigma(i)}(\tau_{e_1})}{\lambda_i(\tau_{e_1})} \right),$$

where S_N denotes the set of permutations of $\{1, \dots, N\}$. Then $q(z) = a_0 + a_1 z + \dots + a_m z^m$ for certain coefficients a_k , with $m = N \cdot N!$, and $a_0 \neq 0$ because all eigenvalues $\lambda_i(\tau_{e_1})$ are different from 0 (since τ_{e_1} is injective). Moreover, $q(z)$ is a multiple of the characteristic polynomial of the operator $\tau_{(P-I)e_1}$, for every $P \in \mathbf{O}(d)$. It follows that $q(z)$ is a multiple of the characteristic

polynomial of τ_z for every $z \in \mathbb{R}^d$ whose norm is equal to the norm of $(P - I)e_1$ for some $P \in \mathbf{O}(d)$, since $\|z_1\| = \|z_2\|$ implies that there exist $P \in \mathbf{O}(d)$ such that $Pz_1 = z_2$, and τ_{z_1}, τ_{Pz_1} have the very same characteristic polynomial. Obviously, $\{\|(P - I)e_1\| : P \in \mathbf{O}(d)\} = [0, 2]$, since $d > 1$. Henceforth, Hamilton's theorem implies that $q(\tau_z) = 0$ for all $\|z\| \leq 2$. Now we can apply Theorem 2.2 to conclude that f is, in distributional sense, an ordinary polynomial. \square

Remark 2.4. For $d = 1$ the result is false, since $V = \text{span}\{2^x, 2^{-x}\}$ is invariant by isometries of \mathbb{R} .

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